Journal of Algebra 359 (2012) 69–79

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On Auslander–Reiten components and splitting trace lattices for integral group rings

Shigeto Kawata

Department of Mathematics, Osaka City University, Osaka 558-8585, Japan

ARTICLE INFO

Article history:
Received 4 July 2011
Available online 28 March 2012
Communicated by Michel Broué

Dedicated to Professor Shigeo Koshitani on his 60th birthday

Keywords:
Auslander–Reiten quivers
Representations of finite groups

ABSTRACT

Let G be a finite group and $\mathcal{O}$ a complete discrete valuation ring of characteristic zero with residue class field $k = \mathcal{O}/\pi\mathcal{O}$ of characteristic $p > 0$. Suppose that $\mathcal{O}$ is sufficiently large to satisfy certain conditions and the group ring $\mathcal{O}G$ is of infinite representation type. Let Θ be a connected component of the Auslander–Reiten quiver of $\mathcal{O}G$. We show that if Θ contains an $\mathcal{O}G$-lattice M such that $M/\pi M$ is an indecomposable kG-module and $\mathrm{rank}_{\mathcal{O}} M$ is not divisible by p, then the tree class of Θ is A_∞ and M lies at the end of Θ.

1. Introduction

Let G be a finite group, p a prime number dividing the order of G and $(K, \mathcal{O}, k)$ a p-modular system, that is, K is a complete discrete valuation field of characteristic zero with multiplicative valuation φ and $\mathcal{O}$ is a valuation ring of φ with unique maximal ideal $\pi\mathcal{O}$ and residue class field $k = \mathcal{O}/\pi\mathcal{O}$ of characteristic p. We use R to denote either $\mathcal{O}$ or k. Let B be a block of the group ring RG and $\Gamma(B)$ the Auslander–Reiten quiver of B. For a connected component Θ of $\Gamma(B)$, we denote by Θ_s the stable part of Θ. Webb [19] showed that the tree class of Θ_s is either a Euclidean diagram or one of the infinite trees A_∞, B_∞, C_∞, D_∞ and A_∞^∞ if the defect group of B is not cyclic. In the case where $R = k$ and k is algebraically closed, Erdmann showed that the tree class of Θ_s is A_∞ if B is of wild representation type [9]. It is known that a block of kG is of wild representation type if its defect group is neither cyclic, dihedral, semidihedral nor generalized quaternion. Concerning the representation type of group rings over $\mathcal{O}$, we refer to the table due to Dieterich [8].

Throughout this paper, we assume that $(K, \mathcal{O}, k) > (K', \mathcal{O}', k')$ is an extension of p-modular system satisfying the following two conditions:

E-mail address: kawata@sci.osaka-cu.ac.jp.

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2012.03.012

(♯I) The ramification index of φ over the valuation φ' of K' is greater than two, namely, $\pi'\mathcal{O} \subseteq \pi^3\mathcal{O}$, where π' is a generator of the unique maximal ideal of $\mathcal{O}'$;

(♯II) $k = k'$ and k is algebraically closed.

An RG-lattice X is called a *splitting trace lattice* if the trace map $\mathrm{Tr} : \mathrm{End}_R X \to R$ is a splittable RG-epimorphism. Auslander and Carlson [3] showed that under the assumption (♯II), an indecomposable RG-lattice X is splitting trace lattice if and only if $\mathrm{rank}_R X$ is not divisible by p. See also Benson and Carlson [5]. In this paper, we consider a connected component Θ of $\Gamma(\mathcal{O}G)$ containing a splitting trace $\mathcal{O}G$-lattice M satisfying one of the conditions (A) or (B) mentioned in Section 3. Assuming that $\mathcal{O}G$ is of infinite representation type, we show that the tree class of Θ is A_∞ under the hypotheses (♯I) and (♯II), see Theorems 3.1 and 3.3. In Section 4, we discuss the tensor product of M with the connected component Δ of $\Gamma(\mathcal{O}G)$ containing the trivial $\mathcal{O}G$-lattice $\mathcal{O}_G$. It will be shown that tensoring with M induces a graph isomorphism from Δ onto Θ, see Theorem 4.1. A similar assertion for $R = k$ in Theorem 4.1 was shown in [11, Proposition 3.3].

For the basic facts and terminology used here, see the books of Assem, Simson and Skowroński [1], Auslander, Reiten and Smalø [2], Benson [4] and Nagao and Tsushima [15].

2. Preliminaries

All RG-modules are assumed to be finitely generated right modules. An RG-lattice means an RG-module which is free as an R-module. If L and M are RG-lattices, then $L \otimes_R M$ is an RG-lattice with the operation of G given by $(x \otimes y)g = xg \otimes yg$ for all $x \in L$, $y \in M$ and $g \in G$. Throughout this paper, $L \otimes M$ means $L \otimes_R M$. For a short exact sequence $\mathcal{S} : 0 \to L_1 \to L_2 \to L_3 \to 0$ of RG-lattices, a tensor product sequence $\mathcal{S} \otimes M : 0 \to L_1 \otimes M \to L_2 \otimes M \to L_3 \otimes M \to 0$ is exact since RG-lattices are R-free. We write L^* for the R-dual $\mathrm{Hom}_R(L, R)$ of L. Let X be an $\mathcal{O}G$-lattice. We denote by $\overline{X}$ the factor module $X/\pi X$, so that $\overline{X}$ is regarded as a kG-module. If $\mathcal{S} : 0 \to X_1 \to X_2 \to X_3 \to 0$ is a short exact sequence of $\mathcal{O}G$-lattices, then we have a short exact sequence $\overline{\mathcal{S}} : 0 \to \overline{X_1} \to \overline{X_2} \to \overline{X_3} \to 0$ of kG-modules.

It is known that the Auslander–Reiten translate of $\Gamma(\mathcal{O}G)$ is the Heller operator Ω in the $\mathcal{O}G$-lattice category. See, for example, [16]. For a non-projective indecomposable $\mathcal{O}G$-lattice X, we write $\mathcal{A}(X)$ for the almost split sequence $0 \to \Omega X \to m(X) \to X \to 0$, where we denote by $m(X)$ the middle term of $\mathcal{A}(X)$. $\mathcal{A}(X)$ is constructed as a pullback of the projective cover P_X of X along an almost projective $\mathcal{O}G$-endomorphism ρ of X:

$$\begin{tikzcd}
0 \arrow[r] & \Omega X \arrow[r] \arrow[d, equal] & m(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pull back}"] & X \arrow[r] \arrow[d, "\rho"] & 0 \\
0 \arrow[r] & \Omega X \arrow[r] & P_X \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

Here, an almost projective $\mathcal{O}G$-endomorphism of X is a generator of the simple socle $\mathrm{Soc}(\underline{\mathrm{End}}_{\mathcal{O}G}(X))$ of $\underline{\mathrm{End}}_{\mathcal{O}G}(X) = \mathrm{End}_{\mathcal{O}G}(X)/\mathrm{End}_{\mathcal{O}G}(X)_1^G$, where $\mathrm{End}_{\mathcal{O}G}(X)_1^G$ is the set of all projective $\mathcal{O}G$-endomorphisms of X (see, for example, [18, (34.11) Theorem]). Since $\mathrm{End}_{\mathcal{O}G}(\mathcal{O}_G) = \mathcal{O}\mathrm{id}_{\mathcal{O}_G}$ and $\mathrm{End}_{\mathcal{O}G}(\mathcal{O}_G)_1^G = |G|\mathcal{O}\mathrm{id}_{\mathcal{O}_G}$ for the trivial $\mathcal{O}G$-lattice $\mathcal{O}_G$, $\frac{|G|}{\pi} \cdot \mathrm{id}_{\mathcal{O}_G}$ is almost projective. Consider the tensor product sequence of $\mathcal{A}(\mathcal{O}_G)$ with an absolutely indecomposable $\mathcal{O}G$-lattice X:

$$\begin{tikzcd}
0 \arrow[r] & \Omega\mathcal{O}_G \otimes X \arrow[r] \arrow[d, equal] & m(\mathcal{O}_G) \otimes X \arrow[r] \arrow[d] & \mathcal{O}_G \otimes X \arrow[r] \arrow[d, "\frac{|G|}{\pi}\cdot\mathrm{id}_{\mathcal{O}_G}\otimes\mathrm{id}_X"] & 0 \\
0 \arrow[r] & \Omega\mathcal{O}_G \otimes X \arrow[r] & P_{\mathcal{O}_G} \otimes X \arrow[r] & \mathcal{O}_G \otimes X \arrow[r] & 0
\end{tikzcd}$$

Carlson and Jones showed that $\frac{|G|}{\pi} \cdot \mathrm{id}_X$ is almost projective if and only if $\mathrm{rank}_{\mathcal{O}} X$ is not divisible by p [6, Proposition 4.7], and $\frac{|G|}{\pi} \cdot \mathrm{id}_X$ is projective if $p \mid \mathrm{rank}_{\mathcal{O}} X$. Hence, we have the following fact due to Auslander and Carlson [3] and Benson and Carlson [5].

Proposition 2.1. *(See [3, Theorem 3.6] and [5, Proposition 2.15].) Assume* ($\sharp$II). *Let X be an indecomposable $\mathcal{O}G$-lattice and $\mathcal{A}(\mathcal{O}_G): 0 \to \Omega\mathcal{O}_G \to m(\mathcal{O}_G) \to \mathcal{O}_G \to 0$ the almost split sequence terminating in the trivial $\mathcal{O}G$-lattice $\mathcal{O}_G$.*

(1) *If $\mathrm{rank}_{\mathcal{O}} X$ is not divisible by p, then the tensor product sequence $\mathcal{A}(\mathcal{O}_G) \otimes X : 0 \to \Omega\mathcal{O}_G \otimes X \to m(\mathcal{O}_G) \otimes X \to \mathcal{O}_G \otimes X \to 0$ is written as a direct sum of the almost split sequence $\mathcal{A}(X)$ and some projective $\mathcal{O}G$-lattice I:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega X & \longrightarrow & m(X) & \longrightarrow & X & \longrightarrow & 0 \\ & & \oplus & & \oplus & & & & \\ & & I & = & I & & & & \end{array}$$

(2) *If $\mathrm{rank}_{\mathcal{O}} X$ is divisible by p, then the tensor product sequence $\mathcal{A}(\mathcal{O}_G) \otimes X$ is split.*

Also, we will need the following fact due to Auslander and Carlson [3] and Benson and Carlson [5].

Proposition 2.2. *Assume* ($\sharp$II), *and let X be an indecomposable RG-lattice.*

(1) ([3, Theorems 3.6, 4.7] *and* [5, Theorem 2.1]) *The following are equivalent*:

(i) *X is a splitting trace lattice, namely, the trace map* $\mathrm{Tr} : \mathrm{End}_R X \to R$ *is splittable*;

(ii) $p \nmid \mathrm{rank}_R X$;

(iii) $R_G \mid \mathrm{End}_R(X) \cong X \otimes X^*$.

If X is a splitting trace lattice, then the multiplicity of R_G in $\mathrm{End}_R(X) \cong X \otimes X^*$ is one.

(2) ([3, Corollary 4.3] *and* [5, Proposition 2.2]) *If the R-rank of X is divisible by p, then so are all R-ranks of the indecomposable direct summands of $X \otimes Y$ for any RG-lattice Y.*

(3) ([3, Proposition 4.1] *and* [5, Theorem 2.1]) *Let Y be an indecomposable RG-lattice. If $R_G \mid X \otimes Y^*$, then $Y \cong X$.*

For the proof of the following fact, see [17, Proposition 2.10].

Lemma 2.3. *Let X be a non-projective indecomposable RG-lattice and Q a subgroup of G. Then an almost split sequence $\mathcal{A}(X)$ splits on restriction to Q if and only if Q does not contain a vertex of X.*

For a kG-module M, the kernel Z of the projective cover P_M of M viewed as an $\mathcal{O}G$-module is called the *Heller $\mathcal{O}G$-lattice* of M: $0 \to Z \to P_M \to M \to 0$ (exact).

Lemma 2.4. *Assume* ($\sharp$I) *and* ($\sharp$II). *Suppose that a non-projective indecomposable $\mathcal{O}G$-lattice L is not isomorphic to any Heller $\mathcal{O}G$-lattice of a kG-module. Let $\mathcal{A}(L): 0 \to \Omega L \to m(L) \to L \to 0$ be the almost split sequence terminating in L. Then the reduced short exact sequence $\overline{\mathcal{A}(L)} : 0 \to \Omega L/\pi\Omega L \to m(L)/\pi m(L) \to L/\pi L \to 0$ of kG-modules is split.*

Proof. See [12, Proposition 4.5]. $\square$

Let Z be an indecomposable non-projective Heller $\mathcal{O}G$-lattice of a kG-module M. Then $\mathrm{rank}_{\mathcal{O}} Z = \mathrm{rank}_{\mathcal{O}} P_M$, where P_M is a projective cover of M as $\mathcal{O}G$-modules, and we see that $p \mid \mathrm{rank}_{\mathcal{O}} Z$. Suppose that Z belongs to a block B of infinite representation type, and let Θ be the connected component of $\Gamma(B)$ containing Z. Then, by [13, Theorem], the tree class of Θ is A_∞ and Z lies at the end of Θ. Thus, the first part of the following lemma holds.

Lemma 2.5. *Assume* ($\sharp$I) *and* ($\sharp$II)*, and suppose that a block B of $\mathcal{O}G$ is of infinite representation type. Let Θ be a connected component of $\Gamma(B)$.*

(1) *If Θ contains a Heller $\mathcal{O}G$-lattice, then the tree class of Θ is A_∞ and all $\mathcal{O}$-ranks of the $\mathcal{O}G$-lattices in Θ are divisible by p.*

(2) *If Θ contains an indecomposable $\mathcal{O}G$-lattice whose $\mathcal{O}$-rank is not divisible by p, then Θ contains neither Heller $\mathcal{O}G$-lattices nor projectives.*

Proof. The second part (2) follows by (1) and [13, Remark 5.5(1)]. $\square$

Lemma 2.6. *Assume* ($\sharp$I) *and* ($\sharp$II)*. Let M be an indecomposable $\mathcal{O}G$-lattice and Θ a connected component of $\Gamma(\mathcal{O}G)$ containing M. Suppose that $M/\pi M$ has an indecomposable direct summand W whose vertex is a Sylow p-subgroup P of G and that Θ does not contain any Heller $\mathcal{O}G$-lattice. Then, for any $\mathcal{O}G$-lattice X in Θ, the following hold.*

(1) *$X/\pi X$ has some syzygy of W as a direct summand. In particular, X has P as vertex.*

(2) *The almost split sequence $\mathcal{A}(X)\downarrow_Q$ restricted to any proper subgroup Q of P splits.*

Proof. Let $M = X_1 - X_2 - \cdots - X_n = X$ be a walk in Θ, so that X_i is a direct summand of the middle term of the almost split sequence $\mathcal{A}(X_{i+1})$ or $\mathcal{A}(\Omega^{-1}X_{i+1})$. We proceed by induction on n. Assume that $X_{n-1}/\pi X_{n-1}$ has some syzygy of W as a direct summand. As X_n is not a Heller lattice, $\overline{\mathcal{A}(X_n)}$ splits by Lemma 2.4. Hence we see that some syzygy of W is a direct summand of $X_n/\pi X_n$ and (1) follows. By (1) and Lemma 2.3, (2) follows. $\square$

Choose and fix a non-projective indecomposable kG-module V. For an $\mathcal{O}G$-lattice X, we denote by $\bar{d}_V(X)$ the number of indecomposable direct summands isomorphic to syzygies of V in an indecomposable decomposition of $X/\pi X$.

Also, fix a subgroup Q of G and a non-projective indecomposable $\mathcal{O}Q$-lattice U. Let us denote by $d_{Q,U}(X)$ the number of indecomposable direct summands isomorphic to syzygies of U in an indecomposable decomposition of $X\downarrow_Q$.

Lemma 2.7. *Assume* ($\sharp$I) *and* ($\sharp$II)*. Let Θ be a connected component of $\Gamma(\mathcal{O}G)$ and suppose that Θ does not contain any Heller $\mathcal{O}G$-lattice. Choose and fix an $\mathcal{O}G$-lattice X in Θ.*

(1) *Let V be a non-projective indecomposable kG-module. If V is a direct summand of $X/\pi X$, then $\bar{d}_V : \Theta \to \mathbb{N}$ is an Ω-periodic additive function on Θ. If no syzygy of V appears as a direct summand of $X/\pi X$, then $\bar{d}_V(Y) = 0$ for every $\mathcal{O}G$-lattice Y in Θ.*

(2) *Suppose that all lattices in Θ have a Sylow p-subgroup P of G as vertex. Let Q $(\neq 1)$ be a proper subgroup of P and U a non-projective indecomposable $\mathcal{O}Q$-lattice. If U is a direct summand of $X\downarrow_Q$, then $d_{Q,U} : \Theta \to \mathbb{N}$ is an Ω-periodic additive function on Θ. If no syzygy of U appears as a direct summand of $X\downarrow_Q$, then $d_{Q,U}(Y) = 0$ for every $\mathcal{O}G$-lattice Y in Θ.*

In particular, if the $\mathcal{O}$-rank of X is not divisible by p, then $\bar{d}_V$ (resp. $d_{Q,U}$) is an Ω-periodic additive function on Θ for an indecomposable direct summand V of $X/\pi X$ (resp. an indecomposable direct summand U of $X\downarrow_Q$ whose $\mathcal{O}$-rank is not divisible by p).

Proof. Lemmas 2.4 and 2.3 imply (1) and (2), respectively. If Θ has an $\mathcal{O}G$-lattice X whose $\mathcal{O}$-rank is not divisible by p, then Θ does not contain any Heller $\mathcal{O}G$-lattice by Lemma 2.5. Note that X is not projective and so $X/\pi X$ is projective-free. $\square$

Recall that if X is a non-projective indecomposable $\mathcal{O}G$-lattice, then $m(X)$ denotes the middle term of the almost split sequence $\mathcal{A}(X)$.

Lemma 2.8. *Assume* ($\sharp$I) *and* ($\sharp$II)*. Suppose that a connected component Θ of $\Gamma(\mathcal{O}G)$ is of type $\mathbb{Z}D_\infty$. Let X and X' be non-projective indecomposable $\mathcal{O}G$-lattices lying at the end of Θ such that $m(X) \cong m(X')$. Then we have* $\operatorname{rank}_{\mathcal{O}} X \equiv \operatorname{rank}_{\mathcal{O}} X' \pmod p$.

Proof. Suppose that

$$X/\pi X = \left(\bigoplus_{\lambda\in\Lambda} \bigoplus_{i\in\mathbb{Z}} a_{\lambda.i}\Omega^i W_\lambda \right) \oplus Y$$

and

$$X'/\pi X' = \left(\bigoplus_{\lambda\in\Lambda} \bigoplus_{i\in\mathbb{Z}} a'_{\lambda.i}\Omega^i W_\lambda \right) \oplus Y'$$

are direct decompositions as kG-modules satisfying the following:

(i) W_λ ($\lambda \in \Lambda$) are indecomposable kG-modules with $p \nmid \dim_k W_\lambda$;

(ii) If $\lambda_1 \neq \lambda_2$, then the Ω-orbits $\{\Omega^i W_{\lambda_1}\}_{i\in\mathbb{Z}}$ of W_{λ_1} and $\{\Omega^i W_{\lambda_2}\}_{i\in\mathbb{Z}}$ of W_{λ_2} are distinct;

(iii) All k-dimensions of the indecomposable direct summands of $Y \oplus Y'$ are divisible by p;

(iv) For each $\lambda \in \Lambda$, the multiplicity $a_{\lambda,i}$ of $\Omega^i W_\lambda$ as a direct summand of $X/\pi X$ is zero if $i < 0$.

Note that W_λ ($\lambda \in \Lambda$) are not Ω-periodic: Indeed, if W_λ is Ω-periodic, then a Sylow p-subgroup of G is cyclic or a generalized quaternion 2-group and all the $\mathcal{O}G$-lattices are Ω-periodic. [10] implies that Θ is an infinite tube. (See also [20, Theorem 1].)

Since Θ does not contain any Heller $\mathcal{O}G$-lattice by Lemma 2.5, both $\overline{\mathcal{A}(X)}$ and $\overline{\mathcal{A}(X')}$ are split by Lemma 2.4. Hence we have

$$\begin{aligned} \Omega X/\pi\Omega X \oplus X/\pi X &\cong m(X)/\pi m(X) \\ &\cong m\big(X'\big)/\pi m\big(X'\big) \cong \Omega X'/\pi\Omega X' \oplus X'/\pi X' \end{aligned}$$

and

$$\begin{aligned} &\left(\bigoplus_{\lambda\in\Lambda} \bigoplus_{0\leqslant i} a_{\lambda.i}\Omega^{i+1} W_\lambda \right) \oplus \left(\bigoplus_{\lambda\in\Lambda} \bigoplus_{0\leqslant i} a_{\lambda.i}\Omega^{i} W_\lambda \right) \\ &\quad\cong \left(\bigoplus_{\lambda\in\Lambda} \bigoplus_{0\leqslant i} a'_{\lambda.i}\Omega^{i+1} W_\lambda \right) \oplus \left(\bigoplus_{\lambda\in\Lambda} \bigoplus_{0\leqslant i} a'_{\lambda.i}\Omega^{i} W_\lambda \right). \end{aligned}$$

For each λ, the multiplicity of W_λ as a direct summand of $m(X)/\pi m(X)$ is $a_{\lambda,0}$, and that as a direct summand of $m(X')/\pi m(X')$ is $a'_{\lambda,0}$. Hence, we have $a_{\lambda,0} = a'_{\lambda,0}$ for each λ. Also, considering the multiplicity of $\Omega^i W_\lambda$, we see $a_{\lambda,i-1} + a_{\lambda,i} = a'_{\lambda,i-1} + a'_{\lambda,i}$ for $1 \leqslant i$. Using induction on i, we have $a_{\lambda,i} = a'_{\lambda,i}$ for all $i \in \mathbb{N}$ and $\lambda \in \Lambda$. □

Lemma 2.9. (1) *Let X be an indecomposable RG-lattice whose R-rank is not divisible by p. Suppose that* $\operatorname{rank}_R X = \operatorname{rank}_R \Omega X$. *Then $p = 2$ and a Sylow 2-subgroup P of G is a cyclic group C_2 of order* 2.

(2) *Let Q be a p-group and U an indecomposable $\mathcal{O}Q$-lattice whose $\mathcal{O}$-rank is not divisible by p. Then $U \ncong \Omega U$.*

Proof. (1) Since $\operatorname{rank}_R X = \operatorname{rank}_R \Omega X = s|P| - \operatorname{rank}_R X$ for some integer s, we have $2(\operatorname{rank}_R X) = s|P|$. Since $\operatorname{rank}_R X$ is not divisible by p, we conclude that $p = 2$ and $|P| = 2$.

(2) Assume that $U \cong \Omega U$. By (1), we see that $p = 2$ and Q is a cyclic group C_2 of order 2. From [7, Proposition 3.1], $\mathcal{O}C_2$ is of finite representation type and $U \cong \mathcal{O}_Q$ or $U \cong \Omega\mathcal{O}_Q$ since $2 \nmid \operatorname{rank}_{\mathcal{O}} U$. But this is absurd since $\mathcal{O}_Q \ncong \Omega\mathcal{O}_Q$. □

If Q is a cyclic p-group, then the Ω-periodicity of $\mathcal{O}_Q$ is 2. It is also known that if $p = 2$ and Q is a generalized quaternion 2-group then the Ω-periodicity of $\mathcal{O}_Q$ is 4.

Lemma 2.10. (1) *Suppose that $p = 2$ and Q is a generalized quaternion 2-group. Let U be an indecomposable $\mathcal{O}Q$-lattice of odd $\mathcal{O}$-rank. Then neither $\Omega\mathcal{O}_Q$ nor $\Omega^3\mathcal{O}_Q$ is a direct summand of $U \otimes U^*$, and $\Omega^2\mathcal{O}_Q$ is a direct summand of $U \otimes U^*$ if and only if $U \cong \Omega^2 U$.*

(2) *Let Q be a cyclic p-group and U an indecomposable $\mathcal{O}Q$-lattice whose $\mathcal{O}$-rank is not divisible by p. Then $\Omega\mathcal{O}_Q$ is not a direct summand of $U \otimes U^*$.*

Proof. (1) Note that for an integer t, $\Omega^t\mathcal{O}_Q \mid U \otimes U^*$ if and only if $\Omega^t U \cong U$ by Proposition 2.2(3). In particular, $\Omega^2\mathcal{O}_Q \mid U \otimes U^*$ if and only if $\Omega^2 U \cong U$. Now we claim that $\Omega\mathcal{O}_Q \nmid U \otimes U^*$: Indeed, if $\Omega\mathcal{O}_Q$ is a direct summand of $U \otimes U^*$, then $U \cong \Omega U$, which contradicts Lemma 2.9(2). Likewise, we see that $\Omega^3\mathcal{O}_Q \nmid U \otimes U^*$.

(2) Since $\Omega U \ncong U$ by Lemma 2.9(2), $\Omega\mathcal{O}_Q$ is not a direct summand of $U \otimes U^*$ by Proposition 2.2(3). $\square$

3. Auslander–Reiten components containing splitting trace lattices

In this section, we consider an indecomposable $\mathcal{O}G$-lattice M satisfying one of the following two conditions:

(A) The multiplicity of k_G as a direct summand of $(M/\pi M) \otimes (M/\pi M)^*$ is one.

(B) The multiplicity of $\mathcal{O}_Q$ as a direct summand of $M{\downarrow_Q} \otimes (M{\downarrow_Q})^*$ is one for some proper subgroup Q of a Sylow p-subgroup of G.

Note that if k is algebraically closed, the condition (A) (resp. (B)) is equivalent to the following (A′) (resp. (B′)) by Proposition 2.2(3):

(A′) $M/\pi M$ has an indecomposable decomposition

$$M/\pi M = V \oplus \left(\bigoplus_i W_i\right)$$

as kG-modules, where $p \nmid \dim_k V$ and $p \mid \dim_k W_i$ for all i. (Possibly $\bigoplus_i W_i$ may be 0.)

(B′) For some proper subgroup Q of a Sylow p-subgroup of G, $M{\downarrow_Q}$ has an indecomposable decomposition

$$M{\downarrow_Q} = U \oplus \left(\bigoplus_i W_i\right)$$

as $\mathcal{O}Q$-lattices, where $p \nmid \mathrm{rank}_{\mathcal{O}} U$ and $p \mid \mathrm{rank}_{\mathcal{O}} W_i$ for all i. (Possibly $\bigoplus_i W_i$ may be 0.)

The following theorem is the main result of this section.

Theorem 3.1. *Assume* ($\sharp$I) *and* ($\sharp$II). *Let M be a non-projective indecomposable $\mathcal{O}G$-lattice satisfying the condition* (A), *and let Θ be a connected component of $\Gamma(\mathcal{O}G)$ containing M. Suppose that M belongs to a block of infinite representation type. Then the tree class of Θ is A_∞ and M lies at the end of Θ.*

Note that Θ does not contain any Heller $\mathcal{O}G$-lattice and we see that $\Theta = \Theta_s$ by Lemma 2.5.

In order to prove the above theorem, we need the following lemma.

Lemma 3.2. *Assume* ($\sharp$I) *and* ($\sharp$II). *Suppose that the group ring $\mathcal{O}G$ is of infinite representation type and a connected component Θ of $\Gamma(\mathcal{O}G)$ contains an $\mathcal{O}G$-lattice whose $\mathcal{O}$-rank is not divisible by p. Then the tree class of Θ is not D_∞.*

Proof. Assume to the contrary that the tree class of Θ is D_∞. Then a part of Θ is as follows for some non-projective indecomposable $\mathcal{O}G$-lattices X and X':

$$\begin{tikzcd}
{} \arrow[dr] & & {} \arrow[dr] & & {} \arrow[dr] & & {} \\
& \Omega m(X) \arrow[ur] \arrow[dr] \arrow[ddr] & & m(X) \arrow[ur] \arrow[dr] \arrow[ddr] & & \Omega^{-1}m(X) \arrow[ur] \arrow[dr] \arrow[ddr] & \\
{} \arrow[ur] & & \Omega X \arrow[ur] & & X \arrow[ur] & & {} \\
{} \arrow[uur] & & \Omega X' \arrow[uur] & & X' \arrow[uur] & & {}
\end{tikzcd}$$

where $m(X)$ is isomorphic to the middle term $m(X')$ of $\mathcal{A}(X')$.

First, we claim that both $\operatorname{rank}_{\mathcal{O}} X$ and $\operatorname{rank}_{\mathcal{O}} X'$ are not divisible by p: Indeed, if $\operatorname{rank}_{\mathcal{O}} X$ is divisible by p, then so is $\operatorname{rank}_{\mathcal{O}} X'$ by Lemma 2.8 and hence all $\mathcal{O}$-ranks of the $\mathcal{O}G$-lattices in Θ are divisible by p.

Let $\mathcal{A}(\mathcal{O}_G) : 0 \to \Omega\mathcal{O}_G \to m(\mathcal{O}_G) \to \mathcal{O}_G \to 0$ be the almost split sequence terminating in $\mathcal{O}_G$. By Proposition 2.1, the tensor product sequences $\mathcal{A}(\mathcal{O}_G) \otimes X$ and $\mathcal{A}(\mathcal{O}_G) \otimes X'$ are the almost split sequences $\mathcal{A}(X)$ modulo projectives and $\mathcal{A}(X')$ modulo projectives, respectively. In particular, $m(\mathcal{O}_G) \otimes X \cong m(\mathcal{O}_G) \otimes X'$ (mod projectives), and we have

$$m(\mathcal{O}_G) \otimes X \otimes X^* \cong m(\mathcal{O}_G) \otimes X' \otimes X^* \quad (\text{mod projectives}).$$

Note that $m(\mathcal{O}_G) \otimes X \otimes X^*$ and $m(\mathcal{O}_G) \otimes X' \otimes X^*$ are the middle terms of $\mathcal{A}(\mathcal{O}_G) \otimes X \otimes X^*$ and $\mathcal{A}(\mathcal{O}_G) \otimes X' \otimes X^*$, respectively.

Let Δ be the connected component of $\Gamma(\mathcal{O}G)$ containing the trivial $\mathcal{O}G$-lattice $\mathcal{O}_G$. Then the tree class of Δ is A_∞ and $\mathcal{O}_G$ lies at the end of Δ by our assumption and [14, Theorem 3.1]. Put

$$X \otimes X^* = \mathcal{O}_G \oplus \left(\bigoplus_i L_i\right) \oplus \left(\bigoplus_j L'_j\right) \oplus N$$

where L_i are indecomposable $\mathcal{O}G$-lattices lying in Δ with $p \nmid \operatorname{rank}_{\mathcal{O}} L_i$ and L'_j are indecomposable $\mathcal{O}G$-lattices in Δ with $p \mid \operatorname{rank}_{\mathcal{O}} L'_j$ and N has no indecomposable direct summand lying in Δ. Then, by Proposition 2.1, it follows that

$$m(\mathcal{O}_G) \otimes X \otimes X^* \cong m(\mathcal{O}_G) \oplus \left(\bigoplus_i m(L_i)\right) \oplus \left(\bigoplus_j \bigl(\Omega L'_j \oplus L'_j\bigr)\right) \oplus N'$$

for some $\mathcal{O}G$-lattice N' which does not have any direct summand lying in Δ. Note that L_i are not isomorphic to any syzygy of $\mathcal{O}_G$ by Proposition 2.2 as X is not Ω-periodic. Hence the number of indecomposable direct summands of $m(\mathcal{O}_G) \otimes X \otimes X^*$ lying in Δ is odd. On the other hand, since no syzygy of $\mathcal{O}_G$ is a direct summand of $X' \otimes X^*$, the number of indecomposable direct summands of $m(\mathcal{O}_G) \otimes X' \otimes X^*$ lying in Δ is even, a contradiction. $\square$

Proof of Theorem 3.1. Note that the tree class is one of the infinite trees A_∞, D_∞ and A_∞^∞ by our assumption and [13, Corollary 5.6]. The tree class of Θ is not D_∞ by Lemma 3.2. Assume to the contrary that the tree class of Θ is A_∞^∞. Then any Ω-periodic additive function on Θ takes a constant value by [4, Proposition 4.5.7]. By Lemma 2.7, $\bar{d}_V$ is an Ω-periodic additive function on Θ. As

$\bar{d}_V(M) = 1$, it follows that $\bar{d}_V(X) = 1$ for all indecomposable $\mathcal{O}G$-lattice X in Θ. Also, by Lemma 2.7, if V' is an indecomposable kG-module whose k-dimension is not divisible by p and V' is not isomorphic to any syzygy of V, then it follows that $\bar{d}_{V'}(X) = 0$ for all X in Θ. Therefore, we see that $\mathrm{rank}_{\mathcal{O}} X \equiv \pm \mathrm{rank}_{\mathcal{O}} M \pmod p$ and $p \nmid \mathrm{rank}_{\mathcal{O}} X$ for all $X \in \Theta$. On the other hand, $\mathcal{A}(\mathcal{O}_G) \otimes M$ is $\mathcal{A}(M)$ modulo projectives by Proposition 2.1. However, since $m(\mathcal{O}_G)$ is indecomposable and $p \mid \mathrm{rank}_{\mathcal{O}} m(\mathcal{O}_G)$, all $\mathcal{O}$-ranks of the indecomposable direct summands of $m(M)$ ($\cong m(\mathcal{O}_G) \otimes M$ modulo projectives) are divisible by p from Proposition 2.2(2), a contradiction.

Now, $\bar{d}_V$ is an Ω-periodic additive function on Θ with tree class A_∞. Since $\bar{d}_V(M) = 1$, M must lie at the end of Θ. □

Theorem 3.3. *Assume* (♯I) *and* (♯II). *Let M be a non-projective indecomposable $\mathcal{O}G$-lattice satisfying the condition* (B), *and let Θ be a connected component of $\Gamma(\mathcal{O}G)$ containing M. Suppose that M belongs to a block of infinite representation type. Then the tree class of Θ is A_∞ and M lies at the end of Θ.*

Proof. Considering the restriction to Q, an $\mathcal{O}Q$-lattice U and $d_{Q,U}$ instead of the reduction modulo (π), a kG-module V and $\bar{d}_V$, we see that a similar argument in the proof of Theorem 3.1 is also valid in order to prove Theorem 3.3. □

We close this section with a remark for the case where $p = 2$.

Proposition 3.4. *Assume that $(K, \mathcal{O}, k)$ is a 2-modular system satisfying the hypotheses* (♯I) *and* (♯II). *Let M be a non-projective indecomposable $\mathcal{O}G$-lattice of odd $\mathcal{O}$-rank, and let Θ be a connected component of $\Gamma(\mathcal{O}G)$ containing M. Suppose that M belongs to a block of infinite representation type. Then the tree class of Θ is A_∞.*

Proof. For $X \in \Theta$, let $d(X)$ be the number of indecomposable direct summands of odd k-dimension in an indecomposable decomposition of kG-module $X/\pi X$. Then d is an Ω-periodic additive function on Θ by Lemmas 2.5(2) and 2.7. Note that $d(M)$ is odd.

The tree class of Θ is A_∞ or A_∞^∞ by [13, Corollary 5.6] and Lemma 3.2. Now assume that the tree class of Θ is A_∞^∞. Then d is constant by [4, Proposition 4.5.7] and, in particular, $d(X)$ $(= d(M))$ is odd for any $X \in \Theta$. Hence we see that all $\mathcal{O}$-ranks of the $\mathcal{O}G$-lattices in Θ are odd. However, since the middle term $m(M)$ of $\mathcal{A}(M)$ is isomorphic to $m(\mathcal{O}_G) \otimes M$ modulo projectives by Proposition 2.1(1) and $m(\mathcal{O}_G)$ is an indecomposable $\mathcal{O}G$-lattice of even $\mathcal{O}$-rank, all $\mathcal{O}$-ranks of the indecomposable direct summands of $m(M)$ are even by Proposition 2.2(2), a contradiction. □

4. Tensor products with splitting trace lattices

In this section, we continue to assume the hypotheses (♯I) and (♯II) in the Introduction and to consider an indecomposable $\mathcal{O}G$-lattice M satisfying the condition (A) or (B) mentioned in Section 3. The aim of this section is to show the following.

Theorem 4.1. *Assume* (♯I) *and* (♯II). *Suppose that an indecomposable $\mathcal{O}G$-lattice M satisfies the condition* (A) *or* (B), *and that M belongs to a block of $\mathcal{O}G$ of infinite representation type. Let Δ be the connected component of $\Gamma(\mathcal{O}G)$ containing the trivial $\mathcal{O}G$-lattice $\mathcal{O}_G$ and Θ the connected component of $\Gamma(\mathcal{O}G)$ containing M. Suppose that Δ is of type $\mathbb{Z}A_\infty$. Then tensoring with M induces a graph isomorphism from Δ onto Θ.*

Note that Θ is of type $\mathbb{Z}A_\infty$ since M is not Ω-periodic and both $\mathcal{O}_G$ and M lie at the ends of Δ and Θ, respectively (Theorems 3.1 and 3.3). We prepare some notation in order to prove Theorem 4.1. Let

$$T : \mathcal{O}_G = L_1 \to L_2 \to \cdots \to L_n \to \cdots$$

be a sequence in Δ such that L_n is a direct summand of the middle term $m(L_{n+1})$ of $\mathcal{A}(L_{n+1})$ and L_n lies in the n-th row from the end of Δ ($\Delta = \Delta_s \cong \mathbb{Z}T$). Also, take a sequence

$$T' : M = M_1 \to M_2 \to \cdots \to M_n \to \cdots$$

in Θ such that M_n is a direct summand of the middle term $m(M_{n+1})$ of $\mathcal{A}(M_{n+1})$ and M_n lies in the n-th row from the end of Θ ($\Theta = \Theta_s \cong \mathbb{Z}T'$). Here, $\mathrm{rank}_{\mathcal{O}} L_{2n-1}$ and $\mathrm{rank}_{\mathcal{O}} M_{2n-1}$ ($n \in \mathbb{N}$) are not divisible by p, and $\mathrm{rank}_{\mathcal{O}} L_{2n}$ and $\mathrm{rank}_{\mathcal{O}} M_{2n}$ ($n \in \mathbb{N}$) are divisible by p.

Lemma 4.2. (1) *$L_n/\pi L_n \cong k_G \oplus \Omega^{-1}k_G \oplus \cdots \oplus \Omega^{-n+1}k_G$ for $n \in \mathbb{N}$. For a proper subgroup Q of a Sylow p-subgroup of G, $L_n\downarrow_Q \cong \mathcal{O}_Q \oplus \Omega^{-1}\mathcal{O}_Q \oplus \cdots \oplus \Omega^{-n+1}\mathcal{O}_Q$ for $n \in \mathbb{N}$.*

(2) *If M satisfies the condition* (A)*, then*

$$M_n/\pi M_n \cong V \oplus \left(\bigoplus_i W_i\right) \oplus \Omega^{-1}\left(V \oplus \left(\bigoplus_i W_i\right)\right) \oplus \cdots \oplus \Omega^{-n+1}\left(V \oplus \left(\bigoplus_i W_i\right)\right)$$

for $n \in \mathbb{N}$, where $M/\pi M = V \oplus (\bigoplus_i W_i)$ is an indecomposable decomposition (A$'$).

(3) *If M satisfies the condition* (B)*, then*

$$M_n\downarrow_Q \cong U \oplus \left(\bigoplus_i W_i\right) \oplus \Omega^{-1}\left(U \oplus \left(\bigoplus_i W_i\right)\right) \oplus \cdots \oplus \Omega^{-n+1}\left(U \oplus \left(\bigoplus_i W_i\right)\right)$$

for $n \in \mathbb{N}$, where $M\downarrow_Q = U \oplus (\bigoplus_i W_i)$ is an indecomposable decomposition (B$'$).

Proof. We show the assertion (1) by induction on n. Assume that $L_t/\pi L_t \cong k_G \oplus \Omega^{-1}k_G \oplus \cdots \oplus \Omega^{-t+1}k_G$ and $L_t\downarrow_Q \cong \mathcal{O}_Q \oplus \Omega^{-1}\mathcal{O}_Q \oplus \cdots \oplus \Omega^{-t+1}\mathcal{O}_Q$ for $1 \leqslant t \leqslant n-1$. The middle term of $\mathcal{A}(\Omega^{-1}L_{n-1})$ is isomorphic to $L_n \oplus \Omega^{-1}L_{n-2}$, and both the reduced sequence $\overline{\mathcal{A}(\Omega^{-1}L_{n-1})}$ of kG-modules and the restricted sequence $\mathcal{A}(\Omega^{-1}L_{n-1})\downarrow_Q$ of $\mathcal{O}Q$-lattices split by Lemmas 2.4 and 2.6. This implies that $L_n/\pi L_n \cong k_G \oplus \Omega^{-1}k_G \oplus \cdots \oplus \Omega^{-n+1}k_G$ and $L_n\downarrow_Q \cong \mathcal{O}_Q \oplus \Omega^{-1}\mathcal{O}_Q \oplus \cdots \oplus \Omega^{-n+1}\mathcal{O}_Q$.

A similar argument as above yields the assertions (2) and (3). □

Let $a(RG)$ be the Green ring of the group ring RG and let $a(RG; p)$ be the linear span in $a(RG)$ of the indecomposable RG-lattices whose R-ranks are divisible by p. Note that $a(RG; p)$ is an ideal of $a(RG)$, see [5].

Lemma 4.3. *Suppose that an indecomposable $\mathcal{O}G$-lattice M satisfies the condition* (A) *or* (B)*. Then the following hold for every $n \in \mathbb{N}$.*

(1) *$M_{2n} \nmid L_{2n+1} \otimes M$ and $\Omega^{-1}M_{2n} \nmid L_{2n+1} \otimes M$.*

(2) *$\Omega^{-1}M \nmid L_3 \otimes M$. Moreover, if M satisfies the condition* (A) *and the trivial $\mathcal{O}G$-lattice $\mathcal{O}_G$ is not Ω-periodic, then*

$$\Omega^{-1}M_{2n-1} \nmid L_{2n+1} \otimes M.$$

Proof. (1) First, we consider the case where M satisfies the condition (A). Assume that $L_{2n+1} \otimes M \cong M_{2n} \oplus N$ for some $\mathcal{O}G$-lattice N. Since

$$(L_{2n+1} \otimes M)/\pi(L_{2n+1} \otimes M) \equiv V \oplus \Omega^{-1}V \oplus \Omega^{-2}V \oplus \cdots \oplus \Omega^{-2n}V$$

and

$$M_{2n}/\pi M_{2n} \equiv V \oplus \Omega^{-1}V \oplus \Omega^{-2}V \oplus \cdots \oplus \Omega^{-(2n-1)}V$$

in $a(kG)/a(kG;p)$ by Lemma 4.2, we see that $N/\pi N \equiv \Omega^{-2n}V$ in $a(kG)/a(kG;p)$. Since $\mathcal{O}_G \mid M \otimes M^*$ and $L_{2n+1} \mid L_{2n+1} \otimes M \otimes M^*$, it follows that $L_{2n+1} \mid N \otimes M^*$ as $M_{2n} \in a(\mathcal{O}G;p)$. Since $L_{2n+1}/\pi L_{2n+1} \cong k_G \oplus \Omega^{-1}k_G \oplus \Omega^{-2}k_G \oplus \cdots \oplus \Omega^{-2n}k_G$ by Lemma 4.2(1) and $(N \otimes M^*)/\pi(N \otimes M^*) \equiv \Omega^{-2n}V \otimes V^*$ in $a(kG)/a(kG;p)$, we have

$$k_G \oplus \Omega^{-1}k_G \oplus \Omega^{-2}k_G \oplus \cdots \oplus \Omega^{-2n}k_G \,\Big|\, \Omega^{-2n}V \otimes V^*$$

and in particular, we see that $k_G \mid \Omega^{-1}V \otimes V^*$ and $V \cong \Omega V$ by Proposition 2.2(3). Thus $p = 2$ and a Sylow 2-subgroup of G is a cyclic group of order 2 by Lemma 2.9(1), and $\mathcal{O}G$ is of finite representation type (see, for example, [8]), a contradiction.

For the case where M satisfies the condition (B), consider the restriction to Q and $\mathcal{O}Q$-lattices U and $\mathcal{O}_Q$ instead of the reduction mod (π) and kG-modules V and k_G. Then a similar argument as above yields $U \cong \Omega U$, but this contradicts Lemma 2.9(2).

Also, we have $\Omega^{-1}M_{2n} \nmid L_{2n+1} \otimes M$ analogously to the arguments above.

(2) Assume that $\Omega^{-1}M_{2n-1} \mid L_{2n+1} \otimes M$. Now $\mathcal{O}_G \mid \Omega^{-1}M_{2n-1} \otimes (\Omega^{-1}M_{2n-1})^*$ since $\operatorname{rank}_{\mathcal{O}}\Omega^{-1}M_{2n-1}$ is not divisible by p. Hence we see that

$$L_{2n+1}^* \,\Big|\, M \otimes \left(\Omega^{-1}M_{2n-1}\right)^*$$

by Proposition 2.2(3).

Now, we consider the case where M satisfies the condition (A) and the trivial $\mathcal{O}G$-lattice $\mathcal{O}_G$ is not Ω-periodic. Then

$$\left(M \otimes \left(\Omega^{-1}M_{2n-1}\right)^*\right)/\pi\left(M \otimes \left(\Omega^{-1}M_{2n-1}\right)^*\right) \equiv V \otimes \left(\Omega^{-1}V \oplus \Omega^{-2}V \oplus \cdots \oplus \Omega^{-(2n-1)}V\right)^*$$

in $a(kG)/a(kG;p)$ and $\overline{M \otimes (\Omega^{-1}M_{2n-1})^*}$ has $\Omega k_G \oplus \Omega^2 k_G \oplus \cdots \oplus \Omega^{2n-1}k_G$ as a direct summand but does not have $k_G \oplus \Omega^{2n}k_G$ as a direct summand since V is not Ω-periodic by our assumption. On the other hand, by Lemma 4.2(1), we have $L_{2n+1}^*/\pi L_{2n+1}^* \cong k_G \oplus \Omega k_G \oplus \Omega^2 k_G \oplus \cdots \oplus \Omega^{2n}k_G$. This forces that $k_G \oplus \Omega^{2n}k_G$ is a direct summand of $\overline{M \otimes (\Omega^{-1}M_{2n-1})^*}$, a contradiction.

Next, we consider the case where M satisfies the condition (B) and $n = 1$. Then $(M \otimes (\Omega^{-1}M_1)^*){\downarrow_Q} \equiv U \otimes (\Omega^{-1}U)^*$ in $a(\mathcal{O}Q)/a(\mathcal{O}Q;p)$. By Lemma 4.2(1), $L_3^*{\downarrow_Q} \cong \mathcal{O}_Q \oplus \Omega\mathcal{O}_Q \oplus \Omega^2\mathcal{O}_Q$. Hence $\mathcal{O}_Q$ is a direct summand of $U \otimes (\Omega^{-1}U)^*$. However, this forces $U \cong \Omega U$ by Proposition 2.2(3), which contradicts Lemma 2.9(2). □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. For $n \in \mathbb{N}$, we shall show the following assertions (1), (2), (3) and (4) by induction on n.

(1) $\Omega^{-1}M_{2n-3} \nmid L_{2n-1} \otimes M$ $(n \geqslant 2)$.

(2) $L_{2n-1} \otimes M \cong M_{2n-1}$ modulo projectives.

(3) $\mathcal{A}(L_{2n-1}) \otimes M = \mathcal{A}(M_{2n-1})$ modulo projectives.

(4) $L_{2n} \otimes M \cong M_{2n}$ modulo projectives.

If $n = 1$, these hold by Proposition 2.1. Assume that the assertions hold for $n-1$. Let $L_{2n-1} \otimes M = (\bigoplus_i N_i) \oplus (\bigoplus_j Y_j)$ be an indecomposable decomposition with $p \nmid \operatorname{rank}_{\mathcal{O}}N_i$ and $p \mid \operatorname{rank}_{\mathcal{O}}Y_j$. Then, since $\mathcal{A}(L_{2n-1}) \otimes M = \mathcal{A}(\mathcal{O}_G) \otimes L_{2n-1} \otimes M$ modulo projectives by Proposition 2.1(1), we have

$$\mathcal{A}(L_{2n-1}) \otimes M = \left(\bigoplus_i \mathcal{A}(N_i)\right) \oplus \left(\bigoplus_j (0 \to \Omega Y_j \to \Omega Y_j \oplus Y_j \to Y_j \to 0)\ (\text{split})\right)$$

modulo projectives. Note that $M_{2n-2} \cong L_{2n-2} \otimes M$ modulo projectives by the inductive hypothesis, and L_{2n-2} is a direct summand of $m(L_{2n-1})$. By Lemma 4.3(1), $M_{2n-2} \ncong Y_j$ for any j. Hence M_{2n-2} is a direct summand of the middle term $m(N_{i_0})$ of $\mathcal{A}(N_{i_0})$ for some i_0.

Here, we claim that the statement (1) holds. Indeed, if $n = 2$ or M satisfies the condition (A), then (1) follows by Lemma 4.3(2). So we consider the case where M satisfies the condition (B). Assume to the contrary that $\Omega^{-1}M_{2n-3} \mid L_{2n-1} \otimes M$. Then $L^*_{2n-1} \mid M \otimes (\Omega^{-1}M_{2n-3})^*$ by using the same argument in the proof of Lemma 4.3(2). Now, $M_{2n-3} \cong L_{2n-3} \otimes M$ modulo projectives by the inductive hypothesis. Hence we also have $\Omega L^*_{2n-3} \mid M \otimes (\Omega^{-1}M_{2n-3})^*$. As $L^*_{2n-1} \not\cong \Omega L^*_{2n-3}$, we see that

$$L^*_{2n-1} \oplus \Omega L^*_{2n-3} \mid M \otimes \left(\Omega^{-1}M_{2n-3}\right)^*.$$

By Lemma 4.2(1), $L_{2n-1}\downarrow_Q$ has $(2n-1)$ syzygies of $\mathcal{O}_Q$ in its indecomposable decomposition and $L_{2n-3}\downarrow_Q$ has $(2n-3)$ syzygies of $\mathcal{O}_Q$ in its indecomposable decomposition. On the other hand, $U \otimes (M_{2n-3}\downarrow_Q)^*$ has at most $2(2n-3)$ syzygies of $\mathcal{O}_Q$ in its indecomposable decomposition by Lemma 2.10 and so does $(M \otimes (\Omega^{-1}M_{2n-3})^*)\downarrow_Q$, a contradiction.

The assertion (1) means that $N_i \not\cong \Omega^{-1}M_{2n-3}$ for all i. Since M_{2n-2} is a direct summand of $m(N_{i_0})$, it follows that $M_{2n-1} \cong N_{i_0}$. As $(L_{2n-1} \otimes M)/\pi(L_{2n-1} \otimes M) \cong M_{2n-1}/\pi M_{2n-1}$ modulo projectives, we conclude that $L_{2n-1} \otimes M \cong M_{2n-1}$ modulo projectives and the assertions (2) and (3) hold. Since L_{2n} is a direct summand of the middle term $m(L_{2n-1})$ of $\mathcal{A}(L_{2n-1})$, the assertion (4) holds. □

Acknowledgments

The author was partially supported by the Japan Society for Promotion of Science, Grant-in-Aid for Scientific Research (C) 23540054. The author would like to thank the referee for carefully reading the manuscript.

References

[1] I. Assem, D. Simson, A. Skowroński, Elements of the Representation Theory of Associative Algebras. 1: Techniques of Representation Theory, London Math. Soc. Stud. Texts, vol. 65, Cambridge Univ. Press, Cambridge, 2006.
[2] M. Auslander, I. Reiten, S. Smalø, Representation Theory of Artin Algebras, Cambridge Stud. Adv. Math., vol. 36, Cambridge Univ. Press, Cambridge, 1995.
[3] M. Auslander, J.F. Carlson, Almost-split sequences and group rings, J. Algebra 103 (1986) 122–140.
[4] D.J. Benson, Representations and Cohomology I, Cambridge Stud. Adv. Math., vol. 30, Cambridge Univ. Press, Cambridge, 1991.
[5] D.J. Benson, J.F. Carlson, Nilpotent elements in the Green ring, J. Algebra 104 (1986) 329–350.
[6] J.F. Carlson, A. Jones, An exponential property of lattices over group rings, J. London Math. Soc. 39 (1989) 467–479.
[7] E. Dieterich, Construction of Auslander–Reiten quivers for a class of group rings, Math. Z. 184 (1983) 43–60.
[8] E. Dieterich, Representation types of group rings over complete discrete valuation rings II, in: I. Reiner, K.W. Roggenkamp (Eds.), Orders and Their Applications, in: Lecture Notes in Math., vol. 1142, Springer, Berlin, 1985, pp. 112–125.
[9] K. Erdmann, On Auslander–Reiten components for group algebras, J. Pure Appl. Algebra 104 (1995) 149–160.
[10] D. Happel, U. Preiser, C.L. Ringel, Vinberg's characterization of Dynkin diagrams using subadditive functions with applications to DTr-periodic modules, in: V. Dlab, P. Gabriel (Eds.), Representation Theory II, in: Lecture Notes in Math., vol. 832, Springer, Berlin, Heidelbelg, New York, 1980, pp. 280–294.
[11] S. Kawata, On Auslander–Reiten components for certain group modules, Osaka J. Math. 30 (1993) 137–157.
[12] S. Kawata, On Heller lattices over ramified extended orders, J. Pure Appl. Algebra 202 (2005) 55–71.
[13] S. Kawata, On Auslander–Reiten components and Heller lattices for integral group rings, Algebr. Represent. Theory 9 (2006) 513–524.
[14] S. Kawata, On Auslander–Reiten components and trivial source lattices for integral group rings, J. Algebra 322 (2009) 1395–1405.
[15] H. Nagao, Y. Tsushima, Representations of Finite Groups, Academic Press, Boston, 1989.
[16] K.W. Roggenkamp, The construction of almost split sequences for integral group rings and orders, Comm. Algebra 5 (1977) 1363–1373.
[17] K.W. Roggenkamp, Integral Representations and Structure of Finite Group Rings, Sém. Math. Sup., vol. 71, Presses de l'Université de Montréal, Montréal, 1980.
[18] J. Thévenaz, G-Algebras and Modular Representation Theory, Clarendon Press, Oxford, 1995.
[19] P.J. Webb, The Auslander–Reiten quiver of a finite group, Math. Z. 179 (1982) 97–121.
[20] A. Wiedemann, Brauer–Thrall I for orders and its application to orders with loops in their Auslander–Reiten graph, in: M. Auslander, E. Lluis (Eds.), Representations of Algebras, Proceedings, Puebla, Mexico, 1980, in: Lecture Notes in Math., vol. 903, Springer, Berlin, Heidelberg, New York, 1981, pp. 350–357.